Journal of Statistical Physics, Vol. 117, Nos. 1/2, October 2004 (© 2004)

Description of Specifications by Means of Probability Distributions in Small Volumes under Condition of Very Weak Positivity

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Received July 18, 2003; accepted April 28, 2004

The problem of description of specifications by means of probability distributions in small volumes with infinite boundary conditions is considered. The description of specifications by means of n-specifications (consistent systems of probability distributions in volumes of cardinality bounded by n with infinite boundary conditions) is established under the condition of very weak positivity. Particular attention is paid to the most important case $n = 1$ which requires special considerations.

KEY WORDS: Consistency conditions; specification; n-specification; positivity conditions; weak positivity; very weak positivity.

1. INTRODUCTION

The notion of specification—consistent system of probability distributions in finite volumes with infinite boundary conditions—is a basic one in the theory of random fields and in mathematical statistical physics. The importance of this notion is that the description of random fields in terms of specifications turned out to be a powerful tool for the development of the theory of random fields (see, for example, ref. 1). Besides, the specifications admitting Gibbsian description represent the mathematical

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background for the study of systems of statistical physics. The problem of Gibbsian description of specifications was a subject of consideration of many authors (see, for example, refs. 2–5).

The theory of description of random fields by means of specifications was constructed by Dobrushin in his fundamental works (refs. 6–8). Particularly, the conditions of existence and uniqueness of random fields described by a given specification were obtained in ref. 6.

In the latter work, while commenting the uniqueness condition, Dobrushin touched upon the problem of restoration of specifications by means of their one-point elements. Several years ago, in a private conversation with one of the authors Dobrushin pointed out the importance of a closely related problem: the problem of description of specifications by means of consistent systems of one-point probability distributions with infinite boundary conditions. However, at that time no consistency conditions on one-point probability distributions were known.

These two problems of Dobrushin were solved by the authors in refs. 9 and 10 under the condition of weak positivity (as well as under the condition of strict positivity). In particular, consistency conditions under which a system of one-point probability distributions with infinite boundary conditions describes a specification were established in ref. 10 under the condition of weak positivity. There it was also shown that the weak positivity condition is coordinating, that is, a specification is weakly positive if and only if its subsystem consisting of one-point elements is weakly positive. It was equally proved that under the condition of weak positivity, the quasilocality property is heritable, that is, a weakly positive specification is quasilocal if and only if its subsystem consisting of one-point elements is quasilocal.

Let us note here that the consistency conditions established in refs. 9 and 10 were mentioned as properties of strictly positive conditional probabilities of Markov random fields in ref. 11. Note also that some results concerning the problem of restoration of strictly positive specifications can be found in refs. 1 and 5. In ref. 12 the attempt to solve the problems of restoration and description of specifications in some non-positive cases was undertaken, but sufficiently full results were obtained only in one-dimensional case and under more complicated conditions.

In the present work the results of ref. 10 are extended to essentially more general situation. First, instead of consistent systems of one-point probability distributions with infinite boundary conditions we consider more general systems: so-called n-specifications, that is, consistent systems of probability distributions in small volumes (volumes of cardinality bounded by n) with infinite boundary conditions. But the principal difference is that the results are obtained under so-called very weak positivity condition which is essentially weaker than the conditions used in ref. 10.

Note that the results of the present work allow one to formulate the condition of existence of random fields described by a given specification in terms of the latter's one-point elements only, that is, exactly in the same terms as the well-known Dobrushin's uniqueness condition. So, it becomes possible to formulate the problem of description of random fields directly in terms of 1-specifications.

Note in addition, that the results of the present work will be probably useful in the recently emerged theory of non-Gibbsian random fields which are now intensively studied (see, for example, ref. 13).

Note finally, that the methods used in the present work are new and considerably differ from those used in ref. 10.

2. PRELIMINARIES

We denote by $\mathbb{Z}^\nu$ the ν-dimensional integer lattice and by $\mathscr{E}$ the set of all finite subsets of $\mathbb{Z}^\nu$, that is, $\mathscr{E}=\{\Lambda\subset\mathbb{Z}^\nu : |\Lambda|<\infty\}$, where $|\Lambda|$ is the cardinality (the number of points) of the set Λ. For convenience of notations, we will omit braces for one-point sets, that is, will write a instead of $\{a\}$. For any $n\in\mathbb{N}\cup\infty=\{1,2,\dots,\infty\}$ we equally denote $\mathscr{E}_n=\{\Lambda\in\mathscr{E} : |\Lambda|\leqslant n\}$. Clearly, for $n=\infty$ we have $\mathscr{E}_\infty=\mathscr{E}$.

Let $(\mathscr{X},\mathscr{F})$ be some measurable *state space*. Usually $\mathscr{X}$ is assumed to be endowed with some topology $\mathscr{T}$, and $\mathscr{F}$ is assumed to be the Borel σ-algebra for this topology. In the present work we concentrate on the case when $\mathscr{X}$ is finite, $\mathscr{T}$ is the discrete topology and $\mathscr{F}$ is the total σ-algebra, that is, $\mathscr{F}=\mathscr{T}=\mathrm{part}(\mathscr{X})$.

For any $T\subset\mathbb{Z}^\nu$ we consider the space $\mathscr{X}^T$ of all configurations on T. For $T=\emptyset$ we assume that $\mathscr{X}^\emptyset=\{\boldsymbol{\emptyset}\}$, where $\boldsymbol{\emptyset}$ is understood as an empty configuration. For any $T,S\subset\mathbb{Z}^\nu$ such that $T\subset S$ and any configuration $\boldsymbol{x}=\{x_t,\ t\in S\}$ on S we denote $\boldsymbol{x}_T$ the *subconfiguration (restriction)* of $\boldsymbol{x}$ on T defined by $\boldsymbol{x}_T=\{x_t,\ t\in T\}$. For any $T,S\subset\mathbb{Z}^\nu$ such that $T\cap S=\emptyset$ and any configurations $\boldsymbol{x}$ on T and $\boldsymbol{y}$ on S we denote $\boldsymbol{xy}$ the *concatenation* of $\boldsymbol{x}$ and $\boldsymbol{y}$, that is, the configuration on $T\cup S$ equal to $\boldsymbol{x}$ on T and to $\boldsymbol{y}$ on S. For any $a\in\mathscr{X}$, $T\subset\mathbb{Z}^\nu$ and $\boldsymbol{x}\in\mathscr{X}^T$, the notation $\boldsymbol{x}\equiv a$ will mean $x_t=a$ for any $t\in T$, and the notation $\boldsymbol{x}\ni a$ will mean $x_t=a$ for some $t\in T$.

Let $\Lambda\in\mathscr{E}$. We denote a probability distribution $\{\mathbf{P}_\Lambda(\boldsymbol{x}),\ \boldsymbol{x}\in\mathscr{X}^\Lambda\}$ on $\mathscr{X}^\Lambda$ by $\mathbf{P}_\Lambda$. Note that in the case $\Lambda=\emptyset$ there exists only one probability distribution defined by $\mathbf{P}_\emptyset(\boldsymbol{\emptyset})=1$. For any $I\subset\Lambda$ we denote $(\mathbf{P}_\Lambda)_I$ the restriction (or marginal distribution) of $\mathbf{P}_\Lambda$ on I, defined by

$$(\mathbf{P}_\Lambda)_I(\boldsymbol{x}) = \sum_{\boldsymbol{y} \in \mathscr{X}^{\Lambda \setminus I}} \mathbf{P}_\Lambda(\boldsymbol{x}\boldsymbol{y}).$$

Finally, let us recall Dobrushin's consistency condition and the notion of specification, introduced in ref. 6.

Definition 1. Let $\Lambda \in \mathscr{E}$. Any system $\{\mathbf{Q}_\Lambda^{\bar{\boldsymbol{x}}},\ \bar{\boldsymbol{x}} \in \mathscr{X}^{\mathbb{Z}^\nu \setminus \Lambda}\}$ of probability distributions on $\mathscr{X}^\Lambda$ indexed by infinite boundary conditions will be called Λ-*kernel* and denoted by $\mathbf{Q}_\Lambda^\bullet$.

Definition 2. Let $\Lambda \in \mathscr{E}$ and $I \subset \Lambda$. We will say that a Λ-kernel $\mathbf{Q}_\Lambda^\bullet$ is *consistent in Dobrushin's sense* with an I-kernel $\mathbf{Q}_I^\bullet$ (and vice versa), if

$$\mathbf{Q}_\Lambda^{\bar{\boldsymbol{x}}}(\boldsymbol{x}\boldsymbol{y}) = (\mathbf{Q}_\Lambda^{\bar{\boldsymbol{x}}})_{\Lambda \setminus I}(\boldsymbol{x})\ \mathbf{Q}_I^{\bar{\boldsymbol{x}}\boldsymbol{x}}(\boldsymbol{y})$$

$$\text{for any } \boldsymbol{x} \in \mathscr{X}^{\Lambda \setminus I},\ \boldsymbol{y} \in \mathscr{X}^I \text{ and } \bar{\boldsymbol{x}} \in \mathscr{X}^{\mathbb{Z}^\nu \setminus \Lambda}.$$

Definition 3. A family $\{\mathbf{Q}_\Lambda^\bullet, \Lambda \in \mathscr{E}\}$ of Λ-kernels indexed by $\Lambda \in \mathscr{E}$ will be called *specification*, if $\mathbf{Q}_\Lambda^\bullet$ and $\mathbf{Q}_I^\bullet$ are consistent in Dobrushin's sense for any $\Lambda \in \mathscr{E}$ and $I \subset \Lambda$.

The main goal of this work is the description of specifications by means of probability distributions in small volumes with infinite boundary conditions, more precisely, by means of n-specifications.

3. NOTION OF *n*-SPECIFICATION AND POSITIVITY POINTS

Recall that specifications are families of Λ-kernels in finite volumes. Let us consider smaller systems: families of Λ-kernels in volumes with bounded size.

Definition 4. Let $n \in \mathbb{N}$. Any family $\{\mathbf{Q}_\Lambda^\bullet, \Lambda \in \mathscr{E}_n\}$ of Λ-kernels indexed by $\Lambda \in \mathscr{E}_n$ will be called n-*system*.

In order to describe specifications, n-systems must satisfy some consistency conditions which should at least be properties of n-systems contained in specifications. So, let us introduce the following notion of n-specification.

Definition 5. Let $n \in \mathbb{N} \setminus 1$. An n-system $\{\mathbf{Q}_\Lambda^\bullet, \Lambda \in \mathscr{E}_n\}$ will be called n-*specification*, if $\mathbf{Q}_\Lambda^\bullet$ and $\mathbf{Q}_I^\bullet$ are consistent in Dobrushin's sense for any $\Lambda \in \mathscr{E}_n$ and $I \subset \Lambda$.

Note that the n-systems contained in specifications are indeed n-specifications. Note also, that in Definitions 4 and 5 one can include the case $n=\infty$, and that ∞-specifications defined this way will be clearly nothing else but specifications.

Remark equally, that we did not yet define the 1-specifications, which would be the most interesting for our purpose, since they are the smallest. Why we did not do it? The matter is that if we introduce the notion of 1-specification in the way of Definition 5, then it would be degenerate, since for 1-systems Dobrushin's consistency conditions become identities. So, in order to define the notion of 1-specification, it is necessary to find some "internal consistency conditions" (that is, some relations between one-point probabilities only), which should be properties of 1-systems contained in specifications. Such properties are given in Theorem 8, but before formulating it let us introduce the notion of positivity point, which will play an important role all along this paper.

Definition 6. Let $\Lambda\in\mathscr{E}$, let $T\subset\mathbb{Z}^\nu\backslash\Lambda$ and $\overline{\boldsymbol{x}}\in\mathscr{X}^{\mathbb{Z}^\nu\setminus\Lambda\setminus T}$, and let $\mathbf{Q}_\Lambda^\bullet$ be a Λ-kernel. A configuration $\boldsymbol{u}\in\mathscr{X}^\Lambda$ is called *positivity point (p.p.) of* $\mathbf{Q}_\Lambda^\bullet$ *under boundary condition (b.c.) varying on T and equal to $\overline{\boldsymbol{x}}$ outside*, if for any $\boldsymbol{\alpha}\in\mathscr{X}^T$, we have $\mathbf{Q}_\Lambda^{\overline{\boldsymbol{x}}\boldsymbol{\alpha}}(\boldsymbol{u})>0$.

Let us formulate immediately one of the most important properties of positivity points.

Theorem 7. Let $J,I\in\mathscr{E}$ such that $J\cap I=\emptyset$, put $\Lambda=J\cup I$, let $T\subset\mathbb{Z}^\nu\setminus\Lambda$ and $\overline{\boldsymbol{x}}\in\mathscr{X}^{\mathbb{Z}^\nu\setminus\Lambda\setminus T}$, and let $\mathbf{Q}_J^\bullet$, $\mathbf{Q}_I^\bullet$ and $\mathbf{Q}_\Lambda^\bullet$ be a J-kernel, an I-kernel and a Λ-kernel. Suppose $\mathbf{Q}_\Lambda^\bullet$ is consistent in Dobrushin's sense both with $\mathbf{Q}_J^\bullet$ and $\mathbf{Q}_I^\bullet$. If $\boldsymbol{u}$ is a p.p. of $\mathbf{Q}_J^\bullet$ under b.c. varying on $I\cup T$ and equal to $\overline{\boldsymbol{x}}$ outside, $\boldsymbol{v}$ is a p.p. of $\mathbf{Q}_I^\bullet$ under b.c. varying on $J\cup T$ and equal to $\overline{\boldsymbol{x}}$ outside, then the concatenation $\boldsymbol{uv}$ is a p.p. of $\mathbf{Q}_\Lambda^\bullet$ under b.c. varying on T and equal to $\overline{\boldsymbol{x}}$ outside.

This theorem will be proved in Section 6, as well as the following theorem presenting the above mentioned properties of 1-systems contained in specifications.

Theorem 8. If $\mathcal{Q}=\{\mathbf{Q}_\Lambda^\bullet,\ \Lambda\in\mathscr{E}_2\}$ is 2-specification, then

$$\mathbf{Q}_t^{\overline{\boldsymbol{x}}v}(x)\ \mathbf{Q}_s^{\overline{\boldsymbol{x}}x}(y)\ \mathbf{Q}_t^{\overline{\boldsymbol{x}}y}(u)\ \mathbf{Q}_s^{\overline{\boldsymbol{x}}u}(v)=\mathbf{Q}_s^{\overline{\boldsymbol{x}}u}(y)\ \mathbf{Q}_t^{\overline{\boldsymbol{x}}y}(x)\ \mathbf{Q}_s^{\overline{\boldsymbol{x}}x}(v)\ \mathbf{Q}_t^{\overline{\boldsymbol{x}}v}(u)$$
$$\text{for any } t,s\in\mathbb{Z}^\nu,\ \ x\in\mathscr{X}^t,\ y,v\in\mathscr{X}^s \text{ and } \overline{\boldsymbol{x}}\in\mathscr{X}^{\mathbb{Z}^\nu\setminus t\setminus s}, \tag{1}$$

and for any p.p. u of $\mathbf{Q}_t^\bullet$ under b.c. varying on s and equal to $\overline{\boldsymbol{x}}$ outside.

Remarks: (1) This theorem remains valid if any one of x, y, u, v is supposed to be a positivity point.

(2) In the formulation of the theorem we could take $\mathcal{Q}$ to be n-specification for some $n \in (\mathbb{N} \setminus 1) \cup \infty$.

(3) In this theorem $\mathcal{Q}$ is arbitrary, and the conditions are imposed on the arguments of the relation (1) only. A weaker version of the theorem was already established by the authors in ref. 10 under some additional conditions on $\mathcal{Q}$. Note also, that it is not possible to obtain the relation (1) without any condition at all. Indeed, as shows the following example this relation may not hold in general.

Example 9. Let the state space $\mathcal{X} = \{0, 1, 2, 3\}$ and consider the ∞-system $\mathcal{Q} = \{\mathbf{Q}_\Lambda^\bullet, \Lambda \in \mathcal{E}\}$ defined by

$$\mathbf{Q}_\Lambda^{\overline{\boldsymbol{x}}}(\boldsymbol{x}) = \begin{cases} \mathbb{1}_{\{\boldsymbol{x} \equiv 0\}} & & \text{if } |\Lambda| \geqslant 2 \\ \mathbb{1}_{\{\boldsymbol{x}=0\}} & \text{if } \overline{\boldsymbol{x}} \ni 0 & \\ 1/5 & \text{if } \overline{\boldsymbol{x}} \equiv 1 \text{ and } \boldsymbol{x} \in \{0,1,2\} & \\ 2/5 & \text{if } \overline{\boldsymbol{x}} \equiv 1 \text{ and } \boldsymbol{x} = 3 & \\ 1/4 & \text{if } \overline{\boldsymbol{x}} \not\ni 0 \text{ and } \overline{\boldsymbol{x}} \not\equiv 1 & \text{if } |\Lambda| = 1. \end{cases}$$

It is not difficult to verify that $\mathcal{Q}$ is a specification. Further, if for some arbitrary $t, s \in \mathbb{Z}^\nu$, we take $\overline{\boldsymbol{x}} \in \mathcal{X}^{\mathbb{Z}^\nu \setminus s \setminus t}$ such that $\overline{\boldsymbol{x}} \equiv 1$, and put $x = 2$, $u = 3$, $y = 1$ and $v = 2$, the relation (1) will clearly fail.

Now, in view of Theorem 8 we can introduce the following notion of 1-specification.

Definition 10. A 1-system $\{\mathbf{Q}_\Lambda^\bullet, \Lambda \in \mathcal{E}_1\}$ is called 1-*specification*, if

$$\mathbf{Q}_t^{\overline{\boldsymbol{x}}v}(x)\, \mathbf{Q}_s^{\overline{\boldsymbol{x}}x}(y)\, \mathbf{Q}_t^{\overline{\boldsymbol{x}}y}(u)\, \mathbf{Q}_s^{\overline{\boldsymbol{x}}u}(v) = \mathbf{Q}_s^{\overline{\boldsymbol{x}}u}(y)\, \mathbf{Q}_t^{\overline{\boldsymbol{x}}y}(x)\, \mathbf{Q}_s^{\overline{\boldsymbol{x}}x}(v)\, \mathbf{Q}_t^{\overline{\boldsymbol{x}}v}(u)$$
$$\text{for any } t, s \in \mathbb{Z}^\nu, x \in \mathcal{X}^t, y, v \in \mathcal{X}^s \text{ and } \overline{\boldsymbol{x}} \in \mathcal{X}^{\mathbb{Z}^\nu \setminus t \setminus s}, \tag{2}$$

and for any p.p. u of $\mathbf{Q}_t^\bullet$ under b.c. varying on s and equal to $\overline{\boldsymbol{x}}$ outside.

Note, that like the case $n \geqslant 2$, the 1-systems contained in specifications will be 1-specifications. So, for any $n \in \mathbb{N}$, the restriction of a specification on $\mathcal{E}_n$ is nothing but an n-specification. Description of specifications by means of n-specifications is in some sense an inverse operation to this restriction.

4. PROBLEM OF DESCRIPTION OF SPECIFICATIONS BY MEANS OF *n*-SPECIFICATIONS

The problems of this type was firstly considered by the authors in refs. 9 and 10. In these works, the problem of description of specifications by means of n-specifications was solved for $n=1$ under the condition of "strict positivity", as well as under the condition of "weak positivity".

4.1. Strict Positivity

The strict positivity is the simplest positivity condition for n-systems.

Definition 11. Let $n \in \mathbb{N} \cup \infty$. An n-system $\{\mathbf{Q}_\Lambda^\bullet, \Lambda \in \mathscr{E}_n\}$ will be called *strictly positive*, if for any $\Lambda \in \mathscr{E}_n$ each configuration $\boldsymbol{x} \in \mathscr{X}^\Lambda$ is a p.p. of $\mathbf{Q}_\Lambda^\bullet$ under b.c. varying on $\mathbb{Z}^\nu \setminus \Lambda$ and equal to $\boldsymbol{\emptyset}$ outside.

Remark that Definition 11 simply means, that for any $\Lambda \in \mathscr{E}_n$, any $\boldsymbol{x} \in \mathscr{X}^\Lambda$ and any $\overline{\boldsymbol{x}} \in \mathscr{X}^{\mathbb{Z}^\nu \setminus \Lambda}$ we have $\mathbf{Q}_\Lambda^{\overline{\boldsymbol{x}}}(\boldsymbol{x}) > 0$.

The strictly positive specifications are widely studied and used in mathematical statistical physics. For example, the specifications admitting Gibbsian description with a real-valued potential are strictly positive.

Note also, that under the condition of strict positivity, the consistency conditions (2) from Definition 10 of 1-specification become

$$\mathbf{Q}_t^{\overline{\boldsymbol{x}}v}(x)\, \mathbf{Q}_s^{\overline{\boldsymbol{x}}x}(y)\, \mathbf{Q}_t^{\overline{\boldsymbol{x}}y}(u)\, \mathbf{Q}_s^{\overline{\boldsymbol{x}}u}(v) = \mathbf{Q}_s^{\overline{\boldsymbol{x}}u}(y)\, \mathbf{Q}_t^{\overline{\boldsymbol{x}}y}(x)\, \mathbf{Q}_s^{\overline{\boldsymbol{x}}x}(v)\, \mathbf{Q}_t^{\overline{\boldsymbol{x}}v}(u)$$
$$\text{for any } t, s \in \mathbb{Z}^\nu, x, u \in \mathscr{X}^t, y, v \in \mathscr{X}^s \text{ and } \overline{\boldsymbol{x}} \in \mathscr{X}^{\mathbb{Z}^\nu \setminus t \setminus s}.$$

Let us now explain the nature and point out several consequences of the problem of description of specifications by means of n-specifications using as example the results obtained in refs. 9 and 10.

The main result is that any strictly positive 1-specification $\boldsymbol{q}$ *describes* a specification, that is, there exists a unique specification containing $\boldsymbol{q}$.

The second result is that the strict positivity condition is *coordinating*, that is, a specification $\boldsymbol{Q}$ is strictly positive if and only if the 1-specification contained in $\boldsymbol{Q}$ is strictly positive. Let us note here that the necessity is trivial, and the sufficiency becomes evident in view of considerations of the present work due to Theorem 7.

Note that these two results imply also that any strictly positive specification $\boldsymbol{Q}$ can be restored by the 1-specification contained in it (that is, any specification containing the same 1-specification is necessarily equal to $\boldsymbol{Q}$) and allow us to conclude that the description is a one-to-one correspondence between strictly positive 1-specifications and strictly positive specifications.

The third result is that under the condition of strict positivity the quasilocality property is *heritable*, that is, a strictly positive specification $\mathcal{Q}$ is quasilocal if and only if the 1-specification contained in $\mathcal{Q}$ is quasilocal.

This result together with the first one allow us to formulate the condition of existence of random fields described by a given specification in terms of the latter's one-point elements only, that is, exactly in the same terms as the well-known Dobrushin's uniqueness condition, and so, it becomes possible to formulate the problem of description of random fields directly in terms of 1-specifications.

Note in addition, that as it will become clear from the subsequent considerations of this work, these results can be extended to the case of arbitrary $n \in \mathbb{N}$.

Now we want to consider the problem of description outside of the scope of strict positivity condition. First of all let us notice that under no condition at all this description does not hold.

4.2. Counterexample

Let us fix some $n \in \mathbb{N}$. If the description of specifications by means of n-specifications held under no condition at all, then any n-specification would describe a specification. The following example shows that it is not true.

Example 12. Let $\mathscr{X} = \{0, 1\}$, denote $F(\boldsymbol{x})$ the function which counts the number of elements equal to 1 in a configuration $\boldsymbol{x}$ on $T \subset \mathbb{Z}^\nu$ and consider the ∞-system $\mathcal{Q} = \{\mathbf{Q}_\Lambda^\bullet, \Lambda \in \mathscr{E}\}$ defined by

$$\mathbf{Q}_\Lambda^{\overline{x}}(\boldsymbol{x}) = \begin{cases} \mathbb{1}_{\{\boldsymbol{x}\equiv 0\}} & \text{if } F(\overline{\boldsymbol{x}}) = 0 \\ \mathbb{1}_{\{\boldsymbol{x}\equiv 1\}} & \text{if } F(\overline{\boldsymbol{x}}) \geqslant 1 \quad \text{if } |\Lambda| \geqslant 2 \\ & \\ \mathbb{1}_{\{\boldsymbol{x}=0\}} & \text{if } F(\overline{\boldsymbol{x}}) = 0 \\ 1/2 & \text{if } F(\overline{\boldsymbol{x}}) = 1 \\ \mathbb{1}_{\{\boldsymbol{x}=1\}} & \text{if } F(\overline{\boldsymbol{x}}) \geqslant 2 \quad \text{if } |\Lambda| = 1. \end{cases}$$

It is not difficult to verify that $\mathcal{Q}$ is a specification. However, the n-specification $\boldsymbol{q}_n$ contained in $\mathcal{Q}$ does not describe a specification, since, for example, the ∞-system $\widehat{\mathcal{Q}} = \{\widehat{\mathbf{Q}}_\Lambda^\bullet, \Lambda \in \mathscr{E}\}$ defined by

$$\widehat{\mathbf{Q}}_\Lambda^{\overline{x}}(\boldsymbol{x}) = \begin{cases} \mathbb{1}_{\{\boldsymbol{x}\equiv 1\}} & \text{if } |\Lambda| \geqslant n+2, \\ \mathbf{Q}_\Lambda^{\overline{x}}(\boldsymbol{x}) & \text{if } |\Lambda| \leqslant n+1 \end{cases}$$

is also a specification containing $\boldsymbol{q}_n$.

So, it becomes evident that in order for the description of specifications by means of n-specifications to hold, some kind of positivity condition is necessary. The strict positivity is the most restrictive positivity condition, since it does not permit zeros at all. A weaker positivity condition is the "weak positivity" which was already studied by the authors in refs. 9 and 10.

4.3. Weak Positivity

The weak positivity condition for n-systems is formulated as follows.

Definition 13. Let $n \in \mathbb{N} \cup \infty$. An n-system $\{\mathbf{Q}^{\bullet}_{\Lambda}, \Lambda \in \mathscr{E}_n\}$ will be called *weakly positive*, if there exist some element $\theta \in \mathscr{X}$ (called *vacuum*), such that for any $\Lambda \in \mathscr{E}_n$ the configuration $\boldsymbol{x} \equiv \theta$ is a p.p. of $\mathbf{Q}^{\bullet}_{\Lambda}$ under b.c. varying on $\mathbb{Z}^{\nu} \backslash \Lambda$ and equal to $\boldsymbol{\emptyset}$ outside.

Clearly, this condition on n-systems is really weaker than the strict positivity one. It remains really weaker when applied to n-specifications too. For instance, the n-specification contained in the specification $\boldsymbol{Q}$ from Example 9 is weakly positive but not strictly positive.

Weakly positive specifications are well known in mathematical statistical physics. For example, the specifications admitting Gibbsian description with a vacuum potential (which may take infinite values) are weakly positive.

Note also, that under the condition of weak positivity, the consistency conditions (2) from Definition 10 of 1-specification have a simpler equivalent form given in the following proposition. The proof of this proposition is quite similar to those of Proposition 18 (see Section 6) and will be omitted.

Proposition 14. A weakly positive 1-system $\{\mathbf{Q}^{\bullet}_{\Lambda}, \Lambda \in \mathscr{E}_1\}$ is 1-specification if and only if

$$\mathbf{Q}_t^{\overline{\boldsymbol{x}}v^\circ}(x)\, \mathbf{Q}_s^{\overline{\boldsymbol{x}}x}(y)\, \mathbf{Q}_t^{\overline{\boldsymbol{x}}y}(u^\circ)\, \mathbf{Q}_s^{\overline{\boldsymbol{x}}u^\circ}(v^\circ) = \mathbf{Q}_s^{\overline{\boldsymbol{x}}u^\circ}(y)\, \mathbf{Q}_t^{\overline{\boldsymbol{x}}y}(x)\, \mathbf{Q}_s^{\overline{\boldsymbol{x}}x}(v^\circ)\, \mathbf{Q}_t^{\overline{\boldsymbol{x}}v^\circ}(u^\circ)$$

$$\text{for any } t, s \in \mathbb{Z}^{\nu}, x \in \mathscr{X}^t, y \in \mathscr{X}^s, \overline{\boldsymbol{x}} \in \mathscr{X}^{\mathbb{Z}^{\nu} \backslash t \backslash s},$$

$$\text{and for } u^\circ \in \mathscr{X}^t \text{ such that } u^\circ = \theta \text{ and } v^\circ \in \mathscr{X}^s \text{ such that } v^\circ = \theta.$$

As we have already mentioned, the problem of description of specifications by means of n-specifications under the condition of weak positivity was solved for $n = 1$ in refs. 9 and 10. There it was shown, that any weakly positive 1-specification describes a specification. It was equally shown, that the weak positivity condition is coordinating, and under this condition the quasilocality property is heritable. Moreover, as it will become clear from the subsequent considerations of this work, these results can be extended to the case of arbitrary $n \in \mathbb{N}$.

So, the further study of the problem of description of specifications by means of n-specifications reduces to determination of a weaker (in ideal case the weakest) positivity condition, under which this description holds. Such a condition is the very weak positivity condition obtained in the present work.

4.4. Very Weak Positivity

Since the positivity points used in Definition 10 of 1-specification are positivity points under boundary condition varying on one-point sets only, it seems natural to consider the following positivity condition.

Definition 15. Let $n \in \mathbb{N} \cup \infty$. An n-system $\{\mathbf{Q}^{\bullet}_{\Lambda}, \Lambda \in \mathscr{E}_n\}$ will be called *too weakly positive*, if for any $\Lambda \in \mathscr{E}_n$, any $s \in \mathbb{Z}^{\nu} \setminus \Lambda$ and any $\overline{\boldsymbol{x}} \in \mathscr{X}^{\mathbb{Z}^{\nu} \setminus \Lambda \setminus s}$, there exists a p.p. of $\mathbf{Q}^{\bullet}_{\Lambda}$ under b.c. varying on s and equal to $\overline{\boldsymbol{x}}$ outside.

However, in accordance with its name, this condition is too weak in order to solve the problem of description. Indeed, a too weakly positive n-specification not necessarily describes a specification (for $n = 1$ it is sufficient to consider the 1-specification $\boldsymbol{q}_1$ from Example 12, and a similar example can be easily constructed for arbitrary $n \in \mathbb{N}$). Moreover, the too weak positivity condition is not coordinating (for instance, the specification $\boldsymbol{Q}$ from Example 12 is not too weakly positive). But what is the matter?

The weak positivity and strict positivity conditions were shown to be coordinating by concatenating positivity points thanks to Theorem 7. But for the too weak positivity condition this approach does not work: if we concatenate two positivity points under boundary conditions varying on one-point sets, we obtain a positivity point under fixed (varying on the empty set) boundary condition. So, we need to modify (strengthen) the condition of too weak positivity in order to be able to correctly concatenate positivity points. This leads us to introduce the following positivity condition.

Definition 16. Let $n \in \mathbb{N} \cup \infty$. An n-system $\{\mathbf{Q}^{\bullet}_{\Lambda}, \Lambda \in \mathscr{E}_n\}$ will be called *very weakly positive*, if for any $\Lambda \in \mathscr{E}_n$, any $V \in \mathscr{E}$ such that $V \subset \mathbb{Z}^{\nu} \setminus \Lambda$ and any $\overline{\boldsymbol{x}} \in \mathscr{X}^{\mathbb{Z}^{\nu} \setminus \Lambda \setminus V}$, there exists some p.p. $\boldsymbol{u} = \theta(\Lambda, V, \overline{\boldsymbol{x}})$ of $\mathbf{Q}^{\bullet}_{\Lambda}$ under b.c. varying on V and equal to $\overline{\boldsymbol{x}}$ outside.

Clearly, this condition on n-systems is really weaker than the weak positivity one. As shows the following example, it remains really weaker when applied to n-specifications too.

Example 17. Let $\mathscr{X}=\{0,1\}$, let F be the function used in Example 12 and consider the ∞-system $\boldsymbol{Q}=\{\mathbf{Q}_\Lambda^\bullet, \Lambda\in\mathscr{E}\}$ defined by

$$\mathbf{Q}_\Lambda^{\overline{\boldsymbol{x}}}(\boldsymbol{x})=\begin{cases}\mathbb{1}_{\{\boldsymbol{x}\equiv 0\}} & \text{if } F(\overline{\boldsymbol{x}})=\infty,\\ \mathbb{1}_{\{\boldsymbol{x}\equiv 1\}} & \text{if } F(\overline{\boldsymbol{x}})<\infty.\end{cases}$$

It is not difficult to verify that $\boldsymbol{Q}$ is a specification, and that the n-specification contained in $\boldsymbol{Q}$ is very weakly positive but not weakly positive.

Note also, that as well as in the weakly positive case, under the condition of very weak positivity, the consistency conditions (2) from Definition 10 of 1-specification have a simpler equivalent form given in the following proposition which will be proved in Section 6.

Proposition 18. A very weakly positive 1-system $\{\mathbf{Q}_\Lambda^\bullet, \Lambda\in\mathscr{E}_1\}$ is 1-specification if and only if

$$\begin{gathered}\mathbf{Q}_t^{\overline{\boldsymbol{x}}v^\circ}(x)\;\mathbf{Q}_s^{\overline{\boldsymbol{x}}x}(y)\;\mathbf{Q}_t^{\overline{\boldsymbol{x}}y}(u^\circ)\;\mathbf{Q}_s^{\overline{\boldsymbol{x}}u^\circ}(v^\circ)=\mathbf{Q}_s^{\overline{\boldsymbol{x}}u^\circ}(y)\;\mathbf{Q}_t^{\overline{\boldsymbol{x}}y}(x)\;\mathbf{Q}_s^{\overline{\boldsymbol{x}}x}(v^\circ)\;\mathbf{Q}_t^{\overline{\boldsymbol{x}}v^\circ}(u^\circ)\\ \text{for any } t,s\in\mathbb{Z}^\nu,\, x\in\mathscr{X}^t,\, y\in\mathscr{X}^s,\, \overline{\boldsymbol{x}}\in\mathscr{X}^{\mathbb{Z}^\nu\setminus t\setminus s}, \qquad (3)\\ \text{and for } u^\circ=\theta(t,s,\overline{\boldsymbol{x}}) \text{ and } v^\circ=\theta(s,t,\overline{\boldsymbol{x}}).\end{gathered}$$

In Section 5 we present the main results of this paper which establish the description of specifications by means of n-specifications under the condition of very weak positivity.

5. MAIN RESULTS AND THEIR PROOFS

The main results of this work consist of the following three theorems.

The first one is that any very weakly positive n-specification describes a specification.

Theorem 19. Let $n\in\mathbb{N}$, and let $\boldsymbol{q}$ be a very weakly positive n-specification. Then there exists a unique specification containing $\boldsymbol{q}$.

The second one is that the very weak positivity condition is coordinating.

Theorem 20. Let $n\in\mathbb{N}$, let $\mathcal{Q}$ be a specification, and let $\boldsymbol{q}$ be the n-specification contained in $\mathcal{Q}$. Then $\mathcal{Q}$ is very weakly positive if and only if $\boldsymbol{q}$ is very weakly positive.

The third one is that under the condition of very weak positivity, the quasilocality property is heritable.

Theorem 21. Let $n\in\mathbb{N}$, let $\mathcal{Q}$ be a very weakly positive specification, and let $\boldsymbol{q}$ be the n-specification contained in $\mathcal{Q}$. Then $\mathcal{Q}$ is quasilocal if and only if $\boldsymbol{q}$ is quasilocal.

The proof of the second theorem is evident, since the necessity is trivial, and the sufficiency directly follows from Theorem 7. The third theorem will become clear in view of the proof of the first one. The proof of the latter will be given in the end of this section and needs some auxiliary results which are of independent interest too. These results are given below and will be proved in Section 6.

Proposition 22. Let $\Lambda\in\mathscr{E}$ and $I\subset\Lambda$. A Λ-kernel $\mathbf{Q}_\Lambda^\bullet$ and an I-kernel $\mathbf{Q}_I^\bullet$ are consistent in Dobrushin's sense if and only if

$$\begin{gathered}\mathbf{Q}_\Lambda^{\overline{\boldsymbol{x}}}(\boldsymbol{x}\boldsymbol{y})\ \mathbf{Q}_I^{\overline{\boldsymbol{x}}\boldsymbol{x}}(\boldsymbol{v})=\mathbf{Q}_\Lambda^{\overline{\boldsymbol{x}}}(\boldsymbol{x}\boldsymbol{v})\ \mathbf{Q}_I^{\overline{\boldsymbol{x}}\boldsymbol{x}}(\boldsymbol{y})\\ \text{for any } \boldsymbol{x}\in\mathscr{X}^{\Lambda\setminus I},\ \boldsymbol{y},\boldsymbol{v}\in\mathscr{X}^I \text{ and } \overline{\boldsymbol{x}}\in\mathscr{X}^{\mathbb{Z}^\nu\setminus\Lambda}.\end{gathered}\tag{4}$$

The equivalent form given in this proposition looks simpler than the original form of Dobrushin's consistency condition and will be intensively used in our considerations.

Proposition 23. Let $n\in(\mathbb{N}\setminus 1)\cup\infty$. An n-system $\mathcal{Q}=\{\mathbf{Q}_\Lambda^\bullet,\ \Lambda\in\mathscr{E}_n\}$ will be n-specification if and only if $\mathbf{Q}_\Lambda^\bullet$ and $\mathbf{Q}_{\Lambda\setminus t}^\bullet$ are consistent in Dobrushin's sense for any $\Lambda\in\mathscr{E}_n$ and $t\in\Lambda$.

This proposition considerably reduces the set of Dobrushin's consistency conditions needed in order to check if an n-system is n-specification.

The next and final theorem establish a general and useful property of n-specifications.

Theorem 24. Let $n\in(\mathbb{N}\setminus 1)\cup\infty$ and let $\mathcal{Q}=\{\mathbf{Q}_\Lambda^\bullet,\ \Lambda\in\mathscr{E}_n\}$ be an n-system.

(1) If $\mathcal{Q}$ is n-specification, then

$$
\begin{gathered}
\mathbf{Q}_A^{\bar{x}u_B}(x_A)\,\mathbf{Q}_B^{\bar{x}x_A}(x_B)\,\mathbf{Q}_C^{\bar{x}x_D}(u_C)\,\mathbf{Q}_D^{\bar{x}u_C}(u_D) \\
= \mathbf{Q}_D^{\bar{x}u_C}(x_D)\,\mathbf{Q}_C^{\bar{x}x_D}(x_C)\,\mathbf{Q}_B^{\bar{x}x_A}(u_B)\,\mathbf{Q}_A^{\bar{x}u_B}(u_A) \\
\text{for any } A, B, C, D \text{ such that} \\
A \cup B = C \cup D \in \mathscr{E}_n \text{ and } A \cap B = C \cap D = \emptyset, \\
\text{and for any } x, u \in \mathscr{X}^{A \cup B} \text{ and } \bar{x} \in \mathscr{X}^{\mathbb{Z}^\nu \setminus A \setminus B} \text{ such that} \\
u_C \text{ is a p.p. of } \mathbf{Q}_C^\bullet \text{ under b.c. varying on } D \text{ and equal to } \bar{x} \text{ outside.}
\end{gathered}
\tag{5}
$$

In particular

$$
\begin{gathered}
\mathbf{Q}_t^{\bar{x}v}(x)\,\mathbf{Q}_{\Lambda\setminus t}^{\bar{x}x}(y)\,\mathbf{Q}_\Lambda^{\bar{x}}(uv) = \mathbf{Q}_\Lambda^{\bar{x}}(xy)\,\mathbf{Q}_{\Lambda\setminus t}^{\bar{x}x}(v)\,\mathbf{Q}_t^{\bar{x}v}(u) \\
\text{for any } \Lambda \in \mathscr{E}_n,\ t \in \Lambda,\ x, u \in \mathscr{X}^t,\ y, v \in \mathscr{X}^{\Lambda\setminus t} \text{ and } \bar{x} \in \mathscr{X}^{\mathbb{Z}^\nu \setminus \Lambda}.
\end{gathered}
\tag{6}
$$

(2) Conversely, if (6) is fulfilled, then $\mathcal{Q}$ is n-specification.

This theorem contains in particular the results of Theorem 8 and at the same time characterizes n-specifications.

Now, we can at last prove the above stated theorem about description of specifications.

Proof of Theorem 19. Let $n \in \mathbb{N}$, and let $q = \{\mathbf{q}_\Lambda^\bullet, \Lambda \in \mathscr{E}_n\}$ be a very weakly positive n-specification.

In order to prove the theorem it is sufficient to show, that there exist a unique $(n+1)$-specification $\mathcal{Q}$ containing q. Indeed, in this case $\mathcal{Q}$ is clearly very weakly positive too, and so we can conclude the proof by means of iteration.

First we prove the uniqueness: if there exists an $(n+1)$-specification $\mathcal{Q} = \{\mathbf{Q}_\Lambda^\bullet, \Lambda \in \mathscr{E}_{n+1}\}$ containing q, then it is the unique $(n+1)$-specification containing q. For each $\Lambda \in \mathscr{E}$ let us fix some point $\ell \in \Lambda$. If $|\Lambda| \leqslant n$, then clearly

$$
\mathbf{Q}_\Lambda^{\bar{x}}(x) = \mathbf{q}_\Lambda^{\bar{x}}(x)\,. \tag{7}
$$

Now let $|\Lambda| = n+1$ and $\bar{x} \in \mathscr{X}^{\mathbb{Z}^\nu \setminus \Lambda}$, and let $u \in \mathscr{X}^\Lambda$ be the configuration defined by $u_t = \theta(t, \Lambda \setminus t, \bar{x})$. Using (6) we have

$$
\mathbf{Q}_\Lambda^{\bar{x}}(x) = \mathbf{Q}_\Lambda^{\bar{x}}(u)\,\frac{\mathbf{q}_\ell^{\bar{x}u_{\Lambda\setminus\ell}}(x_\ell)\,\mathbf{q}_{\Lambda\setminus\ell}^{\bar{x}x_\ell}(x_{\Lambda\setminus\ell})}{\mathbf{q}_\ell^{\bar{x}u_{\Lambda\setminus\ell}}(u_\ell)\,\mathbf{q}_{\Lambda\setminus\ell}^{\bar{x}x_\ell}(u_{\Lambda\setminus\ell})}\,. \tag{8}
$$

Since $\sum_{y\in\mathscr{X}^\Lambda} \mathbf{Q}_\Lambda^{\bar{x}}(y)=1$, we get finally

$$\mathbf{Q}_\Lambda^{\bar{x}}(u)=\left(\sum_{y\in\mathscr{X}^\Lambda} \frac{\mathbf{q}_\ell^{\bar{x}u_{\Lambda\setminus\ell}}(y_\ell)\,\mathbf{q}_{\Lambda\setminus\ell}^{\bar{x}y_\ell}(y_{\Lambda\setminus\ell})}{\mathbf{q}_\ell^{\bar{x}u_{\Lambda\setminus\ell}}(u_\ell)\,\mathbf{q}_{\Lambda\setminus\ell}^{\bar{x}y_\ell}(u_{\Lambda\setminus\ell})}\right)^{-1}. \tag{9}$$

So, any $(n+1)$-specification containing $\boldsymbol{q}$ have necessarily the explicit form given by the formulas (7), (8) and (9), and hence the uniqueness is proved.

To conclude the prove of the theorem, it remains to verify that the $(n+1)$-system $\mathcal{Q}=\{\mathbf{Q}_\Lambda^\bullet,\ \Lambda\in\mathscr{E}_{n+1}\}$ defined by (7), (8) and (9) is indeed an $(n+1)$-specification. Applying Proposition 23 and taking into account that $\boldsymbol{q}$ is n-specification, it is sufficient to verify Dobrushin's consistency condition for $\mathbf{Q}_\Lambda^\bullet$ and $\mathbf{q}_{\Lambda\setminus t}^\bullet$ with $|\Lambda|=n+1$ only. Further, according to Proposition 22 this condition becomes

$$\mathbf{Q}_\Lambda^{\bar{x}}(xy)\,\mathbf{q}_{\Lambda\setminus t}^{\bar{x}x}(v)=\mathbf{Q}_\Lambda^{\bar{x}}(xv)\,\mathbf{q}_{\Lambda\setminus t}^{\bar{x}x}(y). \tag{10}$$

For the case $t=\ell$, using (8) we obtain

$$\begin{aligned}\mathbf{Q}_\Lambda^{\bar{x}}(xy)\,\mathbf{q}_{\Lambda\setminus\ell}^{\bar{x}x}(v) &= \mathbf{Q}_\Lambda^{\bar{x}}(u)\,\frac{\mathbf{q}_\ell^{\bar{x}u_{\Lambda\setminus\ell}}(x)\,\mathbf{q}_{\Lambda\setminus\ell}^{\bar{x}x}(y)}{\mathbf{q}_\ell^{\bar{x}u_{\Lambda\setminus\ell}}(u_\ell)\,\mathbf{q}_{\Lambda\setminus\ell}^{\bar{x}x}(u_{\Lambda\setminus\ell})}\,\mathbf{q}_{\Lambda\setminus\ell}^{\bar{x}x}(v)\\ &= \mathbf{Q}_\Lambda^{\bar{x}}(u)\,\frac{\mathbf{q}_\ell^{\bar{x}u_{\Lambda\setminus\ell}}(x)\,\mathbf{q}_{\Lambda\setminus\ell}^{\bar{x}x}(v)}{\mathbf{q}_\ell^{\bar{x}u_{\Lambda\setminus\ell}}(u_\ell)\,\mathbf{q}_{\Lambda\setminus\ell}^{\bar{x}x}(u_{\Lambda\setminus\ell})}\,\mathbf{q}_{\Lambda\setminus\ell}^{\bar{x}x}(y)\\ &= \mathbf{Q}_\Lambda^{\bar{x}}(xv)\,\mathbf{q}_{\Lambda\setminus\ell}^{\bar{x}x}(y),\end{aligned}$$

and so (10) is verified. Now, for the case of arbitrary $t\in\Lambda$, it is sufficient to show that the right-hand side of (8) does not depend on the choice of ℓ and apply the same argument.

This property is true due to the following chain of equalities

$$\begin{aligned}\frac{\mathbf{q}_\ell^{\bar{x}u_{\Lambda\setminus\ell}}(x_\ell)\,\mathbf{q}_{\Lambda\setminus\ell}^{\bar{x}x_\ell}(x_{\Lambda\setminus\ell})}{\mathbf{q}_\ell^{\bar{x}u_{\Lambda\setminus\ell}}(u_\ell)\,\mathbf{q}_{\Lambda\setminus\ell}^{\bar{x}x_\ell}(u_{\Lambda\setminus\ell})} &= \frac{\mathbf{q}_\ell^{\bar{x}u_{\Lambda\setminus\ell}}(x_\ell)\,\mathbf{q}_t^{\bar{x}x_\ell u_{\Lambda\setminus\ell\setminus t}}(x_t)\,\mathbf{q}_{\Lambda\setminus\ell\setminus t}^{\bar{x}x_\ell x_t}(x_{\Lambda\setminus\ell\setminus t})}{\mathbf{q}_\ell^{\bar{x}u_{\Lambda\setminus\ell}}(u_\ell)\,\mathbf{q}_t^{\bar{x}x_\ell u_{\Lambda\setminus\ell\setminus t}}(u_t)\,\mathbf{q}_{\Lambda\setminus\ell\setminus t}^{\bar{x}x_\ell x_t}(u_{\Lambda\setminus\ell\setminus t})}\\ &= \frac{\mathbf{q}_t^{\bar{x}u_{\Lambda\setminus t}}(x_t)\,\mathbf{q}_\ell^{\bar{x}x_t u_{\Lambda\setminus t\setminus\ell}}(x_\ell)\,\mathbf{q}_{\Lambda\setminus t\setminus\ell}^{\bar{x}x_t x_\ell}(x_{\Lambda\setminus t\setminus\ell})}{\mathbf{q}_t^{\bar{x}u_{\Lambda\setminus t}}(u_t)\,\mathbf{q}_\ell^{\bar{x}x_t u_{\Lambda\setminus t\setminus\ell}}(u_\ell)\,\mathbf{q}_{\Lambda\setminus t\setminus\ell}^{\bar{x}x_t x_\ell}(u_{\Lambda\setminus t\setminus\ell})}\\ &= \frac{\mathbf{q}_t^{\bar{x}u_{\Lambda\setminus t}}(x_t)\,\mathbf{q}_{\Lambda\setminus t}^{\bar{x}x_t}(x_{\Lambda\setminus t})}{\mathbf{q}_t^{\bar{x}u_{\Lambda\setminus t}}(u_t)\,\mathbf{q}_{\Lambda\setminus t}^{\bar{x}x_t}(u_{\Lambda\setminus t})}.\end{aligned}$$

The validity of these equalities in the case $n \geqslant 2$ is guarantied by Theorem 24. For $n=1$ the first and the third equalities are trivial, and the second one follows from the definition of 1-specification. So, the theorem is proved. ▮

6. PROOF OF AUXILIARY RESULTS

Proof of Theorem 7. Let us suppose the contrary: there exists some $\boldsymbol{\alpha} \in \mathscr{X}^T$ such that $\mathbf{Q}_\Lambda^{\overline{\boldsymbol{x}}\boldsymbol{\alpha}}(\boldsymbol{u}\boldsymbol{v})=0$. Since $\mathbf{Q}_\Lambda^\bullet$ is consistent in Dobrushin's sense with $\mathbf{Q}_I^\bullet$, according to Proposition 22 we can write

$$\mathbf{Q}_\Lambda^{\overline{\boldsymbol{x}}\boldsymbol{\alpha}}(\boldsymbol{u}\boldsymbol{v})\ \mathbf{Q}_I^{\overline{\boldsymbol{x}}\boldsymbol{\alpha}\boldsymbol{u}}(\boldsymbol{y}) = \mathbf{Q}_\Lambda^{\overline{\boldsymbol{x}}\boldsymbol{\alpha}}(\boldsymbol{u}\boldsymbol{y})\ \mathbf{Q}_I^{\overline{\boldsymbol{x}}\boldsymbol{\alpha}\boldsymbol{u}}(\boldsymbol{v}).$$

Taking into account that $\boldsymbol{v}$ is a positivity point, we have $\mathbf{Q}_I^{\overline{\boldsymbol{x}}\boldsymbol{\alpha}\boldsymbol{u}}(\boldsymbol{v})>0$, and hence $\mathbf{Q}_\Lambda^{\overline{\boldsymbol{x}}\boldsymbol{\alpha}}(\boldsymbol{u}\boldsymbol{y})=0$ for any $\boldsymbol{y} \in \mathscr{X}^I$.

Similarly, for any $\boldsymbol{y} \in \mathscr{X}^I$, from the relation

$$\mathbf{Q}_\Lambda^{\overline{\boldsymbol{x}}\boldsymbol{\alpha}}(\boldsymbol{u}\boldsymbol{y})\ \mathbf{Q}_J^{\overline{\boldsymbol{x}}\boldsymbol{\alpha}\boldsymbol{y}}(\boldsymbol{x}) = \mathbf{Q}_\Lambda^{\overline{\boldsymbol{x}}\boldsymbol{\alpha}}(\boldsymbol{x}\boldsymbol{y})\ \mathbf{Q}_J^{\overline{\boldsymbol{x}}\boldsymbol{\alpha}\boldsymbol{y}}(\boldsymbol{u}),$$

we get $\mathbf{Q}_\Lambda^{\overline{\boldsymbol{x}}\boldsymbol{\alpha}}(\boldsymbol{x}\boldsymbol{y})=0$ for any $\boldsymbol{x} \in \mathscr{X}^J$.

So $\mathbf{Q}_\Lambda^{\overline{\boldsymbol{x}}\boldsymbol{\alpha}}(z)=0$ for any $z \in \mathscr{X}^\Lambda$, which contradicts the fact that $\mathbf{Q}_\Lambda^{\overline{\boldsymbol{x}}\boldsymbol{\alpha}}$ is a probability distribution. ▮

Proof of Theorem 8. This theorem clearly follows from the first assertion of Theorem 24 by substituting $A=C=t$, $B=D=s$, $\boldsymbol{x}=xy$ and $\boldsymbol{u}=uv$. ▮

Proof of Proposition 18. The necessity is trivial. In order to prove the sufficiency, let us first show that

$$\begin{gathered}\mathbf{Q}_t^{\overline{\boldsymbol{x}}v^\circ}(x)\ \mathbf{Q}_s^{\overline{\boldsymbol{x}}x}(y)\ \mathbf{Q}_t^{\overline{\boldsymbol{x}}y}(u)\ \mathbf{Q}_s^{\overline{\boldsymbol{x}}u}(v^\circ) = \mathbf{Q}_s^{\overline{\boldsymbol{x}}u}(y)\ \mathbf{Q}_t^{\overline{\boldsymbol{x}}y}(x)\ \mathbf{Q}_s^{\overline{\boldsymbol{x}}x}(v^\circ)\ \mathbf{Q}_t^{\overline{\boldsymbol{x}}v^\circ}(u)\\ \text{for any } t,s \in \mathbb{Z}^\nu, x,u \in \mathscr{X}^t, y \in \mathscr{X}^s, \overline{\boldsymbol{x}} \in \mathscr{X}^{\mathbb{Z}^\nu \setminus t \setminus s},\\ \text{and for } v^\circ = \theta(s,t,\overline{\boldsymbol{x}}).\end{gathered} \tag{11}$$

Using (3) we obtain

$$\mathbf{Q}_t^{\overline{\boldsymbol{x}}v^\circ}(x)\ \mathbf{Q}_s^{\overline{\boldsymbol{x}}x}(y)\ \mathbf{Q}_t^{\overline{\boldsymbol{x}}y}(u^\circ)\ \mathbf{Q}_s^{\overline{\boldsymbol{x}}u^\circ}(v^\circ) = \mathbf{Q}_s^{\overline{\boldsymbol{x}}u^\circ}(y)\ \mathbf{Q}_t^{\overline{\boldsymbol{x}}y}(x)\ \mathbf{Q}_s^{\overline{\boldsymbol{x}}x}(v^\circ)\ \mathbf{Q}_t^{\overline{\boldsymbol{x}}v^\circ}(u^\circ),$$

$$\mathbf{Q}_t^{\overline{\boldsymbol{x}}v^\circ}(u)\ \mathbf{Q}_s^{\overline{\boldsymbol{x}}u}(y)\ \mathbf{Q}_t^{\overline{\boldsymbol{x}}y}(u^\circ)\ \mathbf{Q}_s^{\overline{\boldsymbol{x}}u^\circ}(v^\circ) = \mathbf{Q}_s^{\overline{\boldsymbol{x}}u^\circ}(y)\ \mathbf{Q}_t^{\overline{\boldsymbol{x}}y}(u)\ \mathbf{Q}_s^{\overline{\boldsymbol{x}}u}(v^\circ)\ \mathbf{Q}_t^{\overline{\boldsymbol{x}}v^\circ}(u^\circ).$$

Suppose $\mathbf{Q}_s^{\bar{x}u^\circ}(y) > 0$. Then, if we cross-wise multiply these two equalities and cancel identical strictly positive terms, we get the necessary relation. Now suppose $\mathbf{Q}_s^{\bar{x}u^\circ}(y) = 0$. From the same equalities we get clearly $\mathbf{Q}_t^{\bar{x}v^\circ}(x)\,\mathbf{Q}_s^{\bar{x}x}(y) = 0$ and $\mathbf{Q}_t^{\bar{x}v^\circ}(u)\,\mathbf{Q}_s^{\bar{x}u}(y) = 0$, and so the property (11) is proved.

Further, using (11) we obtain

$$\mathbf{Q}_t^{\bar{x}v^\circ}(x)\,\mathbf{Q}_s^{\bar{x}x}(y)\,\mathbf{Q}_t^{\bar{x}y}(u)\,\mathbf{Q}_s^{\bar{x}u}(v^\circ) = \mathbf{Q}_s^{\bar{x}u}(y)\,\mathbf{Q}_t^{\bar{x}y}(x)\,\mathbf{Q}_s^{\bar{x}x}(v^\circ)\,\mathbf{Q}_t^{\bar{x}v^\circ}(u),$$
$$\mathbf{Q}_t^{\bar{x}v^\circ}(x)\,\mathbf{Q}_s^{\bar{x}x}(v)\,\mathbf{Q}_t^{\bar{x}v}(u)\,\mathbf{Q}_s^{\bar{x}u}(v^\circ) = \mathbf{Q}_s^{\bar{x}u}(v)\,\mathbf{Q}_t^{\bar{x}v}(x)\,\mathbf{Q}_s^{\bar{x}x}(v^\circ)\,\mathbf{Q}_t^{\bar{x}v^\circ}(u),$$

and so, applying once more the same argument we can conclude the proof of the proposition. ∎

Proof of Proposition 22. First suppose that $\mathbf{Q}_\Lambda^\bullet$ and $\mathbf{Q}_I^\bullet$ are consistent in Dobrushin's sense. Then

$$\begin{aligned}
\mathbf{Q}_\Lambda^{\bar{x}}(\boldsymbol{x}\boldsymbol{y})\,\mathbf{Q}_I^{\bar{x}\boldsymbol{x}}(\boldsymbol{v}) &= (\mathbf{Q}_\Lambda^{\bar{x}})_{\Lambda\setminus I}(\boldsymbol{x})\,\mathbf{Q}_I^{\bar{x}\boldsymbol{x}}(\boldsymbol{y})\,\mathbf{Q}_I^{\bar{x}\boldsymbol{x}}(\boldsymbol{v})\\
&= (\mathbf{Q}_\Lambda^{\bar{x}})_{\Lambda\setminus I}(\boldsymbol{x})\,\mathbf{Q}_I^{\bar{x}\boldsymbol{x}}(\boldsymbol{v})\,\mathbf{Q}_I^{\bar{x}\boldsymbol{x}}(\boldsymbol{y})\\
&= \mathbf{Q}_\Lambda^{\bar{x}}(\boldsymbol{x}\boldsymbol{v})\,\mathbf{Q}_I^{\bar{x}\boldsymbol{x}}(\boldsymbol{y}),
\end{aligned}$$

and so we have (4).

Now suppose (4). For any $\boldsymbol{v} \in \mathscr{X}^I$ we can write

$$\mathbf{Q}_\Lambda^{\bar{x}}(\boldsymbol{x}\boldsymbol{y})\,\mathbf{Q}_I^{\bar{x}\boldsymbol{x}}(\boldsymbol{v}) = \mathbf{Q}_\Lambda^{\bar{x}}(\boldsymbol{x}\boldsymbol{v})\,\mathbf{Q}_I^{\bar{x}\boldsymbol{x}}(\boldsymbol{y}).$$

Summing over $\boldsymbol{v}$ we obtain

$$\mathbf{Q}_\Lambda^{\bar{x}}(\boldsymbol{x}\boldsymbol{y}) = \sum_{\boldsymbol{v}\in\mathscr{X}^I} \mathbf{Q}_\Lambda^{\bar{x}}(\boldsymbol{x}\boldsymbol{v})\,\mathbf{Q}_I^{\bar{x}\boldsymbol{x}}(\boldsymbol{y}) = (\mathbf{Q}_\Lambda^{\bar{x}})_{\Lambda\setminus I}(\boldsymbol{x})\,\mathbf{Q}_I^{\bar{x}\boldsymbol{x}}(\boldsymbol{y}),$$

and so $\mathbf{Q}_\Lambda^\bullet$ and $\mathbf{Q}_I^\bullet$ are consistent in Dobrushin's sense. ∎

Proof of Proposition 23. The necessity is trivial. In order to prove the sufficiency, it is sufficient to show that the consistency in Dobrushin's sense is transitive, that is, if $J \subset I \subset \Lambda \in \mathscr{E}$, and if a Λ-kernel $\mathbf{Q}_\Lambda^\bullet$ is consistent with an I-kernel $\mathbf{Q}_I^\bullet$ which in its turn is consistent with a J-kernel $\mathbf{Q}_J^\bullet$, then $\mathbf{Q}_\Lambda^\bullet$ and $\mathbf{Q}_J^\bullet$ are also consistent.

Let $\boldsymbol{x} \in \mathscr{X}^{\Lambda \setminus J}$, let $\boldsymbol{y}, \boldsymbol{v} \in \mathscr{X}^J$ and let $\overline{\boldsymbol{x}} \in \mathscr{X}^{\mathbb{Z}^\nu \setminus \Lambda}$. Since $\mathbf{Q}_I^\bullet$ is consistent with $\mathbf{Q}_J^\bullet$, using Proposition 22, we have

$$\mathbf{Q}_I^{\overline{\boldsymbol{x}}\boldsymbol{x}_{\Lambda \setminus I}}(\boldsymbol{x}_{I \setminus J}\boldsymbol{y})\ \mathbf{Q}_J^{\overline{\boldsymbol{x}}\boldsymbol{x}}(\boldsymbol{v}) = \mathbf{Q}_I^{\overline{\boldsymbol{x}}\boldsymbol{x}_{\Lambda \setminus I}}(\boldsymbol{x}_{I \setminus J}\boldsymbol{v})\ \mathbf{Q}_J^{\overline{\boldsymbol{x}}\boldsymbol{x}}(\boldsymbol{y}).$$

Hence

$$\begin{aligned}(\mathbf{Q}_\Lambda^{\overline{\boldsymbol{x}}})_{\Lambda \setminus I}(\boldsymbol{x}_{\Lambda \setminus I})\ \mathbf{Q}_I^{\overline{\boldsymbol{x}}\boldsymbol{x}_{\Lambda \setminus I}}(\boldsymbol{x}_{I \setminus J}\boldsymbol{y})\ \mathbf{Q}_J^{\overline{\boldsymbol{x}}\boldsymbol{x}}(\boldsymbol{v})\\ = (\mathbf{Q}_\Lambda^{\overline{\boldsymbol{x}}})_{\Lambda \setminus I}(\boldsymbol{x}_{\Lambda \setminus I})\ \mathbf{Q}_I^{\overline{\boldsymbol{x}}\boldsymbol{x}_{\Lambda \setminus I}}(\boldsymbol{x}_{I \setminus J}\boldsymbol{v})\ \mathbf{Q}_J^{\overline{\boldsymbol{x}}\boldsymbol{x}}(\boldsymbol{y}).\end{aligned}$$

Further, since $\mathbf{Q}_\Lambda^\bullet$ is consistent with $\mathbf{Q}_I^\bullet$ we obtain

$$\mathbf{Q}_\Lambda^{\overline{\boldsymbol{x}}}(\boldsymbol{x}\boldsymbol{y})\ \mathbf{Q}_J^{\overline{\boldsymbol{x}}\boldsymbol{x}}(\boldsymbol{v}) = \mathbf{Q}_\Lambda^{\overline{\boldsymbol{x}}}(\boldsymbol{x}\boldsymbol{v})\ \mathbf{Q}_J^{\overline{\boldsymbol{x}}\boldsymbol{x}}(\boldsymbol{y}),$$

and so, applying once more Proposition 22 we can conclude the proof of the proposition. ▮

Proof of Theorem 24. In order to carry out the proof we need the following two simple lemmas.

Lemma 25. Let $I, V \in \mathscr{E}$ such that $I \cap V = \emptyset$, put $\Lambda = I \cup V$, let $\overline{\boldsymbol{x}} \in \mathscr{X}^{\mathbb{Z}^\nu \setminus \Lambda}$, and let a Λ-kernel $\mathbf{Q}_\Lambda^\bullet$ be consistent in Dobrushin's sense with an I-kernel $\mathbf{Q}_I^\bullet$. If $\boldsymbol{u}$ is a p.p. of $\mathbf{Q}_I^\bullet$ under b.c. varying on V and equal to $\overline{\boldsymbol{x}}$ outside, then there exists a configuration $\boldsymbol{\gamma} \in \mathscr{X}^V$ such that $\mathbf{Q}_\Lambda^{\overline{\boldsymbol{x}}}(\boldsymbol{u}\boldsymbol{\gamma}) > 0$.

Proof. Let us suppose the contrary: for any configuration $\boldsymbol{\gamma} \in \mathscr{X}^V$ we have $\mathbf{Q}_\Lambda^{\overline{\boldsymbol{x}}}(\boldsymbol{u}\boldsymbol{\gamma}) = 0$. Since $\mathbf{Q}_\Lambda^\bullet$ is consistent in Dobrushin's sense with $\mathbf{Q}_I^\bullet$, for any $\boldsymbol{\alpha} \in \mathscr{X}^I$ and any $\boldsymbol{\gamma} \in \mathscr{X}^V$ according to Proposition 22 we can write

$$\mathbf{Q}_\Lambda^{\overline{\boldsymbol{x}}}(\boldsymbol{u}\boldsymbol{\gamma})\ \mathbf{Q}_I^{\overline{\boldsymbol{x}}\boldsymbol{\gamma}}(\boldsymbol{\alpha}) = \mathbf{Q}_\Lambda^{\overline{\boldsymbol{x}}}(\boldsymbol{\alpha}\boldsymbol{\gamma})\ \mathbf{Q}_I^{\overline{\boldsymbol{x}}\boldsymbol{\gamma}}(\boldsymbol{u}),$$

and hence, taking into account that $\boldsymbol{u}$ is a positivity point we obtain the equality $\mathbf{Q}_\Lambda^{\overline{\boldsymbol{x}}}(\boldsymbol{\alpha}\boldsymbol{\gamma}) = 0$.

So $\mathbf{Q}_\Lambda^{\overline{\boldsymbol{x}}}(\boldsymbol{z}) = 0$ for any $\boldsymbol{z} \in \mathscr{X}^\Lambda$, which contradicts the fact that $\mathbf{Q}_\Lambda^{\overline{\boldsymbol{x}}}$ is a probability distribution. ▮

Lemma 26. Let $\Lambda \in \mathscr{E}$ and $I \subset \Lambda$, let $\overline{\boldsymbol{x}} \in \mathscr{X}^{\mathbb{Z}^\nu \setminus \Lambda}$, and let a Λ-kernel $\mathbf{Q}_\Lambda^\bullet$ be consistent in Dobrushin's sense with an I-kernel $\mathbf{Q}_I^\bullet$. If for some $\boldsymbol{x} \in \mathscr{X}^{\Lambda \setminus I}$ and $\boldsymbol{y}, \boldsymbol{v} \in \mathscr{X}^I$ we have $\mathbf{Q}_\Lambda^{\overline{\boldsymbol{x}}}(\boldsymbol{x}\boldsymbol{y}) = 0$ and $\mathbf{Q}_\Lambda^{\overline{\boldsymbol{x}}}(\boldsymbol{x}\boldsymbol{v}) > 0$, then $\mathbf{Q}_I^{\overline{\boldsymbol{x}}\boldsymbol{x}}(\boldsymbol{y}) = 0$.

Proof. Since $\mathbf{Q}_{\Lambda}^{\bullet}$ is consistent in Dobrushin's sense with $\mathbf{Q}_{I}^{\bullet}$, according to Proposition 22 we can write

$$\mathbf{Q}_{\Lambda}^{\bar{x}}(xy)\ \mathbf{Q}_{I}^{\bar{x}x}(v) = \mathbf{Q}_{\Lambda}^{\bar{x}}(xv)\ \mathbf{Q}_{I}^{\bar{x}x}(y),$$

and so, taking into account that $\mathbf{Q}_{\Lambda}^{\bar{x}}(xy) = 0$ and $\mathbf{Q}_{\Lambda}^{\bar{x}}(xv) > 0$, we obtain immediately $\mathbf{Q}_{I}^{\bar{x}x}(y) = 0$. ∎

Now we turn to the proof of Theorem 24. First let us suppose that Q is n-specification and prove the property (5). For convenience of notations let us denote $\Lambda = A \cup B = C \cup D$. According to Proposition 22, we have

$$\mathbf{Q}_{\Lambda}^{\bar{x}}(x)\ \mathbf{Q}_{B}^{\bar{x}x_A}(u_B) = \mathbf{Q}_{\Lambda}^{\bar{x}}(x_A u_B)\ \mathbf{Q}_{B}^{\bar{x}x_A}(x_B).$$

Multiplying this equality by $\mathbf{Q}_{A}^{\bar{x}u_B}(u_A)$ and using Proposition 22 on the right hand side, we obtain

$$\mathbf{Q}_{\Lambda}^{\bar{x}}(x)\ \mathbf{Q}_{B}^{\bar{x}x_A}(u_B)\ \mathbf{Q}_{A}^{\bar{x}u_B}(u_A) = \mathbf{Q}_{\Lambda}^{\bar{x}}(u)\ \mathbf{Q}_{A}^{\bar{x}u_B}(x_A)\ \mathbf{Q}_{B}^{\bar{x}x_A}(x_B). \tag{12}$$

In the same way we have

$$\mathbf{Q}_{\Lambda}^{\bar{x}}(x)\ \mathbf{Q}_{C}^{\bar{x}x_D}(u_C)\ \mathbf{Q}_{D}^{\bar{x}u_C}(u_D) = \mathbf{Q}_{\Lambda}^{\bar{x}}(u)\ \mathbf{Q}_{D}^{\bar{x}u_C}(x_D)\ \mathbf{Q}_{C}^{\bar{x}x_D}(x_C). \tag{13}$$

Suppose first $\mathbf{Q}_{\Lambda}^{\bar{x}}(x) > 0$ and $\mathbf{Q}_{\Lambda}^{\bar{x}}(u) > 0$. Then, if we cross-wise multiply the equalities (12) and (13) and cancel identical strictly positive terms, we get the relation claimed in (5).

Suppose now $\mathbf{Q}_{\Lambda}^{\bar{x}}(x) = 0$ and $\mathbf{Q}_{\Lambda}^{\bar{x}}(u) > 0$. Then from (12) and (13) we have $\mathbf{Q}_{A}^{\bar{x}u_B}(x_A)\ \mathbf{Q}_{B}^{\bar{x}x_A}(x_B) = 0$ and $\mathbf{Q}_{D}^{\bar{x}u_C}(x_D)\ \mathbf{Q}_{C}^{\bar{x}x_D}(x_C) = 0$ correspondingly, and so, the necessary relation is still valid. Similar considerations show that it remains valid for the case $\mathbf{Q}_{\Lambda}^{\bar{x}}(x) > 0$ and $\mathbf{Q}_{\Lambda}^{\bar{x}}(u) = 0$.

Suppose finally $\mathbf{Q}_{\Lambda}^{\bar{x}}(x) = 0$ and $\mathbf{Q}_{\Lambda}^{\bar{x}}(u) = 0$. Since u_C is a positivity point, due to Lemma 25 there exists some configuration $y \in \mathcal{X}^D$ such that $\mathbf{Q}_{\Lambda}^{\bar{x}}(u_C y) > 0$. The latter inequality together with $\mathbf{Q}_{\Lambda}^{\bar{x}}(u) = 0$ implies according to Lemma 26 that $\mathbf{Q}_{D}^{\bar{x}u_C}(u_D) = 0$, and so, the left hand side of the relation claimed in (5) vanishes. It remains to show that the right hand side of this relation vanishes too. Indeed, if $\mathbf{Q}_{\Lambda}^{\bar{x}}(u_C x_D) = 0$ then taking into consideration that $\mathbf{Q}_{\Lambda}^{\bar{x}}(u_C y) > 0$ and using Lemma 26 we obtain $\mathbf{Q}_{D}^{\bar{x}u_C}(x_D) = 0$, and if $\mathbf{Q}_{\Lambda}^{\bar{x}}(u_C x_D) > 0$ then taking into account that $\mathbf{Q}_{\Lambda}^{\bar{x}}(x) = 0$ we get $\mathbf{Q}_{C}^{\bar{x}x_D}(x_C) = 0$.

So, the property (5) is established. In order to prove (6) it is sufficient now to put $A = t$, $B = \Lambda \setminus t$, $C = \emptyset$, $D = \Lambda$, $\boldsymbol{x} = x\boldsymbol{y}$ and $\boldsymbol{u} = u\boldsymbol{v}$ in (5), and note that $\boldsymbol{u}_\emptyset = \boldsymbol{\emptyset}$ is indeed a p.p. of $\mathbf{Q}_\emptyset^\bullet$ under b.c. varying on Λ and equal to $\overline{\boldsymbol{x}}$ outside.

It remains to prove the second part of the theorem. Suppose (6) is fulfilled, take some $\Lambda \in \mathscr{E}_n$, $t \in \Lambda$, $x \in \mathscr{X}^t$, $\boldsymbol{y}, \boldsymbol{v} \in \mathscr{X}^{\Lambda \setminus t}$ and $\overline{\boldsymbol{x}} \in \mathscr{X}^{\mathbb{Z}^\nu \setminus \Lambda}$, and let us show that

$$\mathbf{Q}_\Lambda^{\overline{\boldsymbol{x}}}(x\boldsymbol{y})\ \mathbf{Q}_{\Lambda \setminus t}^{\overline{\boldsymbol{x}}x}(\boldsymbol{v}) = \mathbf{Q}_\Lambda^{\overline{\boldsymbol{x}}}(x\boldsymbol{v})\ \mathbf{Q}_{\Lambda \setminus t}^{\overline{\boldsymbol{x}}x}(\boldsymbol{y}). \tag{14}$$

Suppose first $\mathbf{Q}_t^{\overline{\boldsymbol{x}}\boldsymbol{v}}(x) > 0$. Then, taking $u = x$ in (6) and canceling the term $\mathbf{Q}_t^{\overline{\boldsymbol{x}}\boldsymbol{v}}(x)$ we obtain (14).

Suppose now $\mathbf{Q}_t^{\overline{\boldsymbol{x}}\boldsymbol{y}}(x) > 0$. Then, interchanging the positions of $\boldsymbol{y}$ and $\boldsymbol{v}$ in (6), taking $u = x$ and canceling the term $\mathbf{Q}_t^{\overline{\boldsymbol{x}}\boldsymbol{y}}(x)$ we obtain (14).

Suppose finally $\mathbf{Q}_t^{\overline{\boldsymbol{x}}\boldsymbol{v}}(x) = 0$ and $\mathbf{Q}_t^{\overline{\boldsymbol{x}}\boldsymbol{y}}(x) = 0$. Taking in consideration the first equality, we can show that the left hand side of the relation (14) vanishes. Indeed, since $\mathbf{Q}_t^{\overline{\boldsymbol{x}}\boldsymbol{v}}$ is probability distribution, we can chose $u \in \mathscr{X}^t$ such that $\mathbf{Q}_t^{\overline{\boldsymbol{x}}\boldsymbol{v}}(u) > 0$, and using (6) we clearly obtain $\mathbf{Q}_\Lambda^{\overline{\boldsymbol{x}}}(x\boldsymbol{y})\ \mathbf{Q}_{\Lambda \setminus t}^{\overline{\boldsymbol{x}}x}(\boldsymbol{v}) = 0$. Similarly, the second equality implies that the right hand side of the relation (14) vanishes, and so this relation is proved.

Now, in order to conclude the proof of the theorem it remains to apply consecutively Propositions 22 and 23. ▌

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